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Stable determination of a simple metric, a covector field and a potential from the hyperbolic Dirichlet-to-Neumann map

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Abstract

Let (M,g) be a compact Riemmanian manifold with non-empty boundary. Consider the second order hyperbolic initial-boundary value problem

$$\begin{cases} (\partial_t^2 + P(x,D))u = 0 & \text{in} \quad (0,T)\times M, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{for} \quad x \in M, \\ u(t,x) = f(t,x) & \text{on} \quad (0,T)\times \partial M, \end{cases}$$

where

$$P(x,D) = -\frac{1}{\sqrt{\det g}}\left(-\frac{\partial}{\partial x^j} + \mathrm{i}\, b_j\right) g^{ij}\sqrt{\det g}\left(-\frac{\partial}{\partial x^i} + \mathrm{i}\, b_i\right) + q$$

is a first-order perturbation of the Laplace operator $-\triangle_g$ on (M,g). Here b and q are a covector field and a potential, respectively, in M. We prove Hölder type stability estimates near generic simple Riemannian metrics for the inverse problem of recovering g, b, and q from the Dirichlet-to-Neumann(DN) map associated, $f \to \partial_\nu u - \mathrm{i}\,\langle \nu, b\rangle_g u|_{\partial M\times[0,T]}$ modulo a class of transformations that fixed the DN map.

1 Introduction and main results

Let (M,g) be an oriented and compact Riemannian manifold with non-empty boundary ∂M. In this paper we consider the stability of the inverse problem of determining a Riemannian metric together with the lower order coefficients of the second order hyperbolic initial-boundary value problem 2, from the information that is encoded in the hyperbolic Dirichlet-to-Neumann(DN) map Λ_g, see (3). The question that we want to address is the following: if two hyperbolic DN maps are close in an appropriate topology, how close are the Riemannian metrics and the lower order coefficients? We now describe in more detailed the problem and the results.

We denote by $\triangle_g$ the Laplace-Beltrami. In local coordinates, $g(x) = (g_{ij}(x))$ and

$$\triangle_g u = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\left(g^{ij}\sqrt{\det g}\frac{\partial u}{\partial x^i}\right)$$

where $(g^{ij}) = (g_{ij})^{-1}$.

Any second order uniformly elliptic operator with real principal part can be written in local coordinates as

$$P(x,D) = -\frac{1}{\sqrt{\det g}}\left(-\frac{\partial}{\partial x^j} + \mathrm{i}\, b_j\right) g^{ij}\sqrt{\det g}\left(-\frac{\partial}{\partial x^i} + \mathrm{i}\, b_i\right) + q \tag{1}$$

where b is a complex-valued covector field on M and q the complex-valued function on M. Moreover, it is self-adjoint w.r.t. $L^2 := L^2(M, dV_g)$ if and only if b and q are real valued, see next section.

For such $P(x,D)$ let us consider the problem

$$\left\{\begin{array}{rcll} (\partial_t^2 + P(x,D))u & = & 0 & \text{in} \quad (0,T)\times M, \\ u(0,x) = \partial_t u(0,x) & = & 0 & \text{for} \quad x \in M, \\ u(t,x) & = & f(t,x) & \text{on} \quad (0,T)\times \partial M, \end{array}\right. \tag{2}$$

where $f \in C_0^1(\mathbb{R}_+ \times \partial M)$. Denote by $\nu = \nu(x)$ the outer unit conormal to ∂M at $x \in \partial M$, normalized so that $g^{ij}\nu_i\nu_j = 1$. The hyperbolic DN map $\Lambda_P : H_0^1([0,T]\times\partial M) \to L^2([0,T]\times\partial M)$ is defined by

$$\Lambda_P(f) := \frac{\partial u}{\partial \nu} - \mathrm{i}\,\langle \nu, b\rangle_g u\Big|_{(0,T)\times\partial M} = \sum_{i=1}^n \nu^i \frac{\partial u}{\partial x^i} - \mathrm{i}\,\nu^i b_i u\,\Big|_{(0,T)\times\partial M}, \tag{3}$$

where $\nu^i = \sum g^{ij}\nu_j$. This modification of the traditional DN map comes naturally from the Green identity

$$(Pu, v)_{L^2(M)} - (u, Pv)_{L^2(M)} = (\Lambda_P u, v)_{L^2(\partial M)} - (u, \Lambda_P v)_{L^2(\partial M)}$$

of (1). This DN map is invariant under the group of transformations

$$g \mapsto g_* := \varphi^* g, \qquad b \mapsto b_* := \varphi^* b - \mathrm{d}\theta \qquad \text{and} \qquad q \mapsto q_* := \varphi^* q, \tag{4}$$

where $\varphi : M \to M$ is a diffeomorphism with $\varphi|_{\partial M} =$Id and $\theta \in C^\infty(M, \mathbb{R})$ with $\theta|_{\partial M} = 0$, see (19).

The inverse problem is therefore formulated in the following way: knowing Λ_P,

1. Can one determine the metric g, the covector field b and the potential q up to a transformation that acts on this coefficients by (4)?
2. Could we do this recovery in an stable way?

The physical interpretation of this problem is to find the speed of the wave propagation (i.e, g) and other physical properties (encoded in b and q) inside an unknown body by making measurements on the boundary. The initial zero conditions in (2) mean that the body is "quiet" initially and we use a disturbance f to recover information about the interior of body; this comes naturally in many applications (e.g. geophysics).

When $b = 0$, the first question was solved in the Euclidean case ($g = e$) by Rakesh and Symes [19] using the result of Sylvester and Uhlmann [28]. For the non Euclidean case, the spectral analogue of the inverse boundary value problem, was solved for the case $b = q = 0$ (see also [7]) by Belishev and Kurylev [5] using the boundary control method introduced by Belishev [3], [4].

This proof uses in a very essential way a unique continuation principle proven by Tataru [29]. Because of the latter, it is unlikely that this method would prove Hölder type of stability estimates even under geometric and topological restrictions.

For the general case, the spectral analogue of this problem as well as some modifications were considered in [17], [1], [10], [9], [22], Isakov [11], Romanov [20], [13] and references in there. The first question has a positive answer in the general case due to Kurylev and Lassas [14], they used a geometrical formulation (see [15]) of the boundary control method introduced in [3].

The problem of stability has presented slower progress. Conditional type of estimates are typical for such kind of inverse problems. We assume an apriori compactness type of condition of boundedness of the C^k norm of the metric, the covector field and the potential for some large k. This guarantee continuity of the inverse map without control of the modulus of continuity. Weak geometric conditions of this type were studied in [2].

For the Euclidean case and when $b = 0$, Sun [27] proved continuity of the inverse problem. Also for the Euclidean case, Isakov and Sun [12] proved logarithmic stability for 2 dimensional domains and Hölder stability in 3 dimensions. For the anisotropic case $b = q = 0$, in [23] Stefanov and Uhlmann proved Hölder stability for metrics close to the euclidean in a C^k norm, later they extended their result of Hölder stability near generic simple metris in [26]. For the case $b = 0$, Dos Santos and Bellassoued [6] proved Hölder stability for the potential q when g_0 is a fixed simple metric, they also recover in a stable way the conormal factor of the metric g_0 within the conormal class when $q = 0$.

In this paper we prove Hölder type of conditional stability when recovering: the metric g, the covector field b and the potential q; all at the same time, under the hypothesis that the metrics are near a generic simple metric. A Riemannian manifold (M, g) is simple if ∂M is strictly convex and any two point in M can be connected by a a single minimizing geodesic depending smoothly on them, see Definition 1 for more details.

Since simple manifolds are diffeomorphic to the unit ball in the Euclidean space, from now on, without loss of generality, we consider the case that $M = \overline{\Omega}$, where Ω is diffeomorphic to a ball in the Euclidean space and has smooth boundary. There exist a dense open subset $\mathcal{G}^k(\Omega)$, of simple metrics in $C^k(\bar{\Omega})$ for $k \gg 1$ that consists of those metrics for which the x-ray transform is s-injective and stable, see definition 3. This set $\mathcal{G}^k(\Omega)$ contains all real analytic simple metrics in M, see [25].

The main result of this paper reads as follows. Let P be the operator related to the metric g, covector field b and potential q as in (1), similarly let $\tilde{P}$ be the operator related to $\tilde{g}$, $\tilde{b}$ and $\tilde{q}$. Suppose that we consider the initial boundary problem (2) for both operators.

Theorem 1. *There exist $k \gg 1$, $0 < \mu < 1$, such that for any $g_0 \in \mathcal{G}^k(\Omega)$ there exist ϵ_0 such that if*

$$\begin{gathered} ||g - g_0||_{C(\bar{\Omega})} + ||\tilde{g} - g_0||_{C(\bar{\Omega})} + ||b - \tilde{b}||_{C(\bar{\Omega})} < \epsilon_0 \\ \textit{and} \\ g, \tilde{g}, b, \tilde{b}, q, \tilde{q} \textit{ belongs to a bounded set in } C^k(\bar{\Omega}), \end{gathered} \tag{5}$$

then the following holds. For $T > diam_{g_0}(\Omega)$ there exist a $C^3(\bar{\Omega})$ diffeomorphism

φ and smooth function θ as in (4), such that

$$||g-\tilde{g}_*||_{C^2(\bar{\Omega})}+||b-\tilde{b}_*||_{C^1(\bar{\Omega})}+||q-\tilde{q}_*||_{C(\bar{\Omega})} \leq C||\Lambda_P-\Lambda_{\tilde{P}}||^{\mu}_{H^1_0([0,\epsilon]\times\partial\Omega)\to L^2([0,T]\times\partial\Omega)} \tag{6}$$

for any $0<\epsilon\leq T$.

Remark 1. All $C^k(\Omega)$ norms are related to an arbitrary but fixed choice of coordinates.

The idea of the proof is to divide the recovery in three parts: First we prove stability at the boundary using asymptotic solutions pointing in different directions to divide the information. Second, we recover the boundary distance function from the data and use boundary rigidity estimates in [25] to obtain stability for the metric. Third we translate the problem of recovering b and q to an x-ray type of problem. We can explicitly recover the x-ray transform along geodesics of b and q from the data. We then use estimates obtained in [24] to get the desired stability. In each step we must separate the information to get estimates for g, b and q separately.

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2 Preliminaries

Let us first state some mapping properties for the initial boundary value problem (2) and some remarks about self adjointness. Any second order uniformly elliptic operator with real principal part can be written as

$$P=-\triangle_g+B+Q \tag{7}$$

where B is a complex-valued vector field which in local coordinates has the form $B=B^j\partial/\partial x^j$ and Q is a complex-valued function on M. An straight calculation shows that P is self adjoint with respect to $L^2(M,dV_g)$ if and only if

$$\operatorname{Re}B=0 \text{ and } \operatorname{Div}B=\mathrm{i}\,2\operatorname{Im}Q. \tag{8}$$

If we write (7) as in (1) then we get that

$$B=\mathrm{i}\,2b^{\sharp} \text{ and } Q=q+|b|_g+\mathrm{i}\ \operatorname{Div}b^{\sharp}. \tag{9}$$

where $(b^{\sharp})^j=g^{ij}b_i$. We then see that by (8) the operator (1) is self adjoint w.r.t L^2 if and only if b and q are real valued.

The direct problem (2) has the following mapping properties, see [16], [11] and [13].

Lemma 2. *Let*

$$F\in L^1([0,T];L^2(M)) \text{ and } f\in H^1_0([0,T]\times\partial M). \tag{10}$$

Then there is a unique solution $u(t,x)$ of the problem

$$\begin{cases} (\partial_t^2+P(x,D))u &= F(t,x) \quad \text{in} \quad (0,T)\times M, \\ u(0,x)=\partial_t u(0,x) &= 0 \quad \text{for} \quad x\in M, \\ u(t,x) &= f(t,x) \quad \text{on} \quad (0,T)\times\partial M, \end{cases} \tag{11}$$

such that

$$u(t,x)\in C([0,T];H^1(M))\cap C^1([0,T];L^2(M))$$

and

$$\max_{0\le t\le T}\{||u(t)||_{H^1(M)}+||u_t(t)||_{L^2(M)}\}\le c(T)\{\int_0^T ||F(t,\cdot)||_{L^2(M)}dt+||f||_{H_0^1([0,T]\times\partial M)}\}.$$

Moreover $\Lambda f|_{[0,T]\times\partial M}\in L^2([0,T]\times\partial M)$ *and*

$$||\Lambda f|_{[0,T]\times\partial M}||_{L^2([0,T]\times\partial M)}\le c(T)\{\int_0^T ||F(t,\cdot)||_{L^2(M)}dt+||f||_{H_0^1([0,T]\times\partial M)}\}.$$

To simplify notation we denote from now on,

$$||\cdot||_*=||\cdot||_{H_0^1([0,\epsilon]\times\partial\Omega)\to L^2([0,T]\times\partial\Omega)}. \tag{12}$$

where ϵ and T are as in Theorem 1. More precisely, if Λ is the DN map defined above then $||\cdot||_*$ is defined as the supremum of $||\Lambda||_{H_0^1([0,T]\times\partial\Omega)}$ over all $f\in H_0^1([0,\epsilon]\times\partial\Omega)$ such that $||f||_{H_0^1([0,\epsilon]\times\partial\Omega)}=1$. This is well defined since one can extend f as zero for $t>\epsilon$ and this extension of f will be in $H_0^1([0,T]\times\partial\Omega)$ with $f|_{t=0}=0$. Moreover, because of uniqueness for (11), for any $0<\epsilon<T$ and $0<T'\le T$ we have

$$||\Lambda_P||_{H_0^1([0,\epsilon]\times\partial\Omega)\to L^2([0,T']\times\partial\Omega)}\le ||\Lambda_P||_{H_0^1([0,T]\times\partial\Omega)\to L^2([0,T]\times\partial\Omega)}. \tag{13}$$

2.1 Group of transformations that keep invariant the DN-map

We will consider the type of transformations that do not change the DN map. Let

$$\varphi: M\to M$$

be a diffeomorphism with $\varphi|_{\partial M}$ =Id, let us denote $x=\varphi(y)$ then for any $v\in C_0^\infty(M)$

$$(P(x,D)u,v)_{L^2}=(\varphi^*P(y,D)\varphi^*u,\varphi^*v)_{L^2}$$

where the operatos φ^*P is as in (1) with

$$\varphi^*g,\varphi^*b \text{ and } \varphi^*q \qquad \text{instead of} \qquad g,b \text{ and } q \tag{14}$$

respectively. Here φ^* denotes the pull-back with respect to the metric g. Since $v\in C_0^\infty(M)$ is arbitrary we get

$$P(x,D)u=0 \iff \varphi^*P(x,D)\varphi^*u=0. \tag{15}$$

Making a change of variable in the DN map operator we see that for any $f\in C_0^1(\mathbb{R}_+\times\partial M)$

$$\begin{aligned}\Lambda_P(f)&=\nu^i\tfrac{\partial u}{\partial x^i}-\mathrm{i}\,\nu^ib_iu\,\Big|_{(0,T)\times\partial M}=\nu^i\tfrac{\partial y^k}{\partial x^i}\tfrac{\partial\varphi^*u}{\partial y^k}-\mathrm{i}\,\nu^i\tfrac{\partial y^k}{\partial x^i}b_i\tfrac{\partial x^i}{\partial y^k}\varphi^*u\,\Big|_{(0,T)\times\partial M}\\&=(\varphi^*\nu)^k\tfrac{\partial\varphi^*u}{\partial y^k}-\mathrm{i}\,(\varphi^*\nu)^k(\varphi^*b)_k\varphi^*u\,\Big|_{(0,T)\times\partial M}=\Lambda_{\varphi^*P}(f)\end{aligned}$$

the last equality follows from (15), hence $\Lambda_{\varphi^* P} = \Lambda_P$.

There is also another type of transformation that keep the DN map invariant, let

$$\theta \in C^\infty(M;\mathbb{R}), \theta|_{\partial M} = 0. \tag{16}$$

We denote P_θ the operator as in (1) where b is replaced by

$$b_\theta = b - \mathrm{d}\theta \tag{17}$$

we claim that $\Lambda_{P_\theta} = \Lambda_P$. Similarly as before we can show that

$$Pu = 0 \iff P_\theta v = 0 \tag{18}$$

where $u = e^{\mathrm{i}\,\theta} v$, then we get that for any $f \in C^1_0(\mathbb{R}_+ \times \partial M)$ since $e^{\mathrm{i}\,\theta}|_{\partial M} = 1$

$$\begin{aligned} \Lambda_P(f) &= \nu^i \tfrac{\partial u}{\partial x^i} - \mathrm{i}\,\nu^i b_i u \,\big|_{(0,T)\times\partial M} = \nu^i \tfrac{\partial e^{\mathrm{i}\,\theta} v}{\partial x^i} - \mathrm{i}\,\nu^i b_i e^{\mathrm{i}\,\theta} v \,\Big|_{(0,T)\times\partial M} \\ &= \nu^i \tfrac{\partial v}{\partial x^i} - \mathrm{i}\,\nu^i (b_i - \tfrac{\partial \theta}{\partial x^i}) v \,\big|_{(0,T)\times\partial M} = \Lambda_{P_\theta}(f) \end{aligned}$$

were the last equation follows by (18). It is worth mention that transformations (14) and (17) commute since $\varphi^* b - \varphi^* \mathrm{d}\theta = \varphi^* b - \mathrm{d}\varphi^*\theta$. Previous discussion justifies the definition of the set of operators that fix Λ_P as:

$$[P(g,b,q)] := \left\{ (\varphi^* g, \varphi^* b - \mathrm{d}\theta, \varphi^* q) : {\scriptstyle \varphi \text{ is a diffeomorphism, } \varphi|_{\partial M} = 1 \atop \theta \in C^\infty(M;\mathbb{C}) \text{ and } \theta|_{\partial M} = 0} \right\} \tag{19}$$

We emphasis that the recovery of g, b and q is done up to this class (19).

2.2 Simple metrics and geodesic x-ray transform

In this section we define the set of generic simple metrics that we will use for the proof of Theorem 1.

Definition 1. We say that g is simple in M, if ∂M is strictly convex w.r.t g, and for each $x \in \bar{M}$, the exponential map $\exp_x : \exp_x^{-1}(\bar{M}) \to \bar{M}$ is a diffeomorphism.

Remark 2. Note that since all requirements for simplicity are open, then a small $C^k(\overline{\Omega})$ perturbation of a simple metric in Ω is also simple, so we can extend g in a strictly convex neighborhood $\Omega_1 \supset \Omega$ as a simple metric in Ω_1.

Consider the Hamiltonian $H_g(x,\xi) = (g^{ij}\xi_i\xi_j)/2$. We denote by $(x(t),\xi(t))$ the corresponding integral curves of H_g on the energy level $H_g = 1/2$. We use the following parametrization of those bicharacteristics. Denote

$$\Gamma_-(\Omega_1) := \{(z,\omega) \in T^*\Omega_1; z \in \partial\Omega_1, |\omega|_g = 1, \langle \omega, \nu(z)\rangle_g < 0\}, \tag{20}$$

where $\nu(z)$ is the outer unit conormal to $\partial\Omega$. Introduce the measure

$$d\mu(z,\omega) = |\langle \omega, \nu(z)\rangle_g| dS_z dS_\omega \quad \text{on } \Gamma_-,$$

where dS_z and dS_ω are the surface measures on $\partial\Omega$ and $\{\omega \in T^*_x\Omega_1; |\omega|_g = 1\}$ in the metric, respectively. Define $(x(t),\xi(t)) = (x(t;z,\omega),\xi(t;z,\omega))$ to be the bicharacteristics issued from $(z,\omega) \in \Gamma_-(\Omega_1)$.

For any covector field $b = b_i dx^i$ we define the geodesic x-ray transform $I_g b$ by

$$(I_g b)(z,\omega) = \int b_i(x(t))\xi^i(t)dt, \qquad (z,\omega) \in \Gamma_-, \tag{21}$$

similarly for any symmetric 2-tensor $f = f_{ij}dx^i dx^j$ the geodesic x-ray transform $I_g f$, which is a linearization of the boundary rigidity problem, is defined as

$$(I_g f)(z,\omega) = \int f_{ij}(x(t))\xi^i(t)\xi^j(t)dt, \qquad (z,\omega) \in \Gamma_-, \tag{22}$$

where $(x(t), \xi(t)) = (x(t; z, \omega), \xi(t; z, \omega))$ as above is the maximal bicharacteristics in Ω_1 issued from (z, ω) and $\dot{x}^i = g_{ij}\xi_j = \xi^i$.

For a vector field v let d^s be the symmetric differential defined by $(\mathrm{d}^s v)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_i v_j)$ where ∇_i are the covariant derivatives. It is known that

$$I_g \mathrm{d}^s v = 0 \text{ for any vector field } v \text{ with } v|_{\partial\Omega} = 0.$$

Definition 2. We say that I_g is s-inyective in Ω, if $Igf = 0$ and $f \in L^2(\Omega)$ imply $f = \mathrm{d}^s v$ for some vector field $v \in H^1_0(\Omega)$.

Definition 3. Given $k \geq 2$, define $\mathcal{G}^k := \mathcal{G}^k(\Omega)$ as the set of all simple $C^k(\overline{\Omega})$ metrics in Ω for with the map I_g is s-injective.

By [25], for $k \gg 1$, $\mathcal{G}^k$ is an open and dense subset of all simple $C^k(\overline{\Omega})$ metrics in Ω, in particular, all real analytic simple metrics belong to $\mathcal{G}^k$. Also all metrics with a small enough bound on the curvature (in particular all negatively curved metrics) belong to $\mathcal{G}^k$, see [21] and references in there.

2.3 Interpolation Estimates

This interpolation result is needed to use the apriori conditions in (5). We fix a simple metric $g_0 \in C^k$ for $k \gg 1$. By remark 2 extend g_0 as a simple metric in some $\Omega_1 \supset \overline{\Omega}$. Let $g, \tilde{g}$ be as in Theorem 1, then there exist $A > 0$ and $\varepsilon_0 \ll 1$ such that

$$||g||_{C^k(\overline{\Omega})} + ||\tilde{g}||_{C^k(\overline{\Omega})} \leq A, \qquad ||g - g_0||_{C(\overline{\Omega})} + ||\tilde{g} - g_0||_{C(\overline{\Omega})} \leq \varepsilon_0. \tag{23}$$

The first condition above is a typical compactness condition. Using the interpolation estimate in [30],

$$||f||_{C^t(\overline{\Omega})} \leq C||f||^{1-\theta}_{C^{t_1}(\overline{\Omega})}||f||^{\theta}_{C^{t_2}(\overline{\Omega})} \tag{24}$$

where $0 < \theta < 1$, $t_1 \geq 0$, $t_2 \geq 0$, one gets that

$$||g - g_0||_{C^t(\overline{\Omega})} \leq C_M \varepsilon_0^{(k-t)/k} \tag{25}$$

for each $t \geq 0$, if $k > t$; the same is true for $\tilde{g}$. For our purposes, it is enough to apply (24) with t, t_1 and t_2 integers only, then (24) extends to compact manifolds with or without boundary.

2.4 Boundary normal coordinates and vector field transformation near the boundary

In this section we will construct a diffeomorphism φ that fixes the boundary, and the smooth function θ that modifies the covector field, so that near the boundary they have convenient properties for calculations. To fix notation we sate the following well known proposition about boundary normal coordinates.

Proposition 3. *Let M be a Riemannian manifold with compact boundary ∂M. Then there exist $T > 0$ and a neighborhood $N \subset M$ of the boundary ∂M together with a diffeomorphism $\varphi : \partial M \times [0, T) \to N$ such that: (1) $\varphi_p(0) = p$ for all $p \in \partial M$ and (2) the unique unit-speed geodesic normal to ∂M starting at any $p \in \partial M$ is given by $t \mapsto \varphi_p(t)$.*

Remark 3. For $t > 0$, $\varphi_p(t) = x$ so that the distance from x to ∂M is equal to t, i.e. $\varphi_p(\mathrm{d}(x, \partial M)) = x$ for all x in the interior of M.

For the two metrics g and $\tilde{g}$, there exist φ_1 and φ_2 diffeomorphisms like in Proposition 3. Then

$$\varphi := \varphi_1 \circ \varphi_2^{-1} \tag{26}$$

is a diffeomorphism near $\partial\Omega$ fixing $\partial\Omega$, and mapping the unit speed geodesics for g normal to $\partial\Omega$ into unit speed geodesics for $\tilde{g}$ normal to $\partial\Omega$. By [18] this diffeomorphism can be extended to a global diffeomorphism, let us called φ again for its extension.

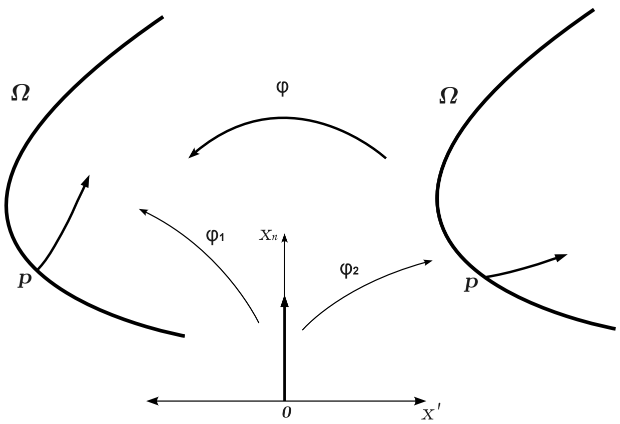

Figure 1: Boundary normal coordinates

Notice that g and $\varphi^*\tilde{g}$ have common normal geodesics to $\partial\Omega$, close to $\partial\Omega$, and moreover, if (x', x^n) are boundary normal coordinates near a fixed boundary point for one of those metrics, they are also boundary normal coordinates for the other metric, see Figure 1.

Now let b, $\tilde{b}$ be two covector fields in Ω as in Theorem (1). Use boundary normal coordinates (x', x^n) near the boundary $\partial\Omega$ and let $b = b_i dx^i$ and $\tilde{b} =$

$\tilde{b}_i dx^i$. Define

$$\theta(x', x^n) = \int_0^{x^n} (\varphi^* \tilde{b}_n - b_n)(x', y) dy. \tag{27}$$

Extend θ so that $\theta \in C^\infty(\Omega; \mathbb{C})$, notice $\theta|_{\partial\Omega} = 0$. If we use this θ as in (4), we get that near the boundary $\partial\Omega$

$$(\tilde{b}_*)_n = (\varphi^* \tilde{b} - \mathrm{d}\theta)_n = b_n. \tag{28}$$

Now by the invariance of the DN map under this type of transformations from now on we will modify the initial coeficients

$$\tilde{g} \mapsto \varphi^* \tilde{g}, \qquad \tilde{b} \mapsto \varphi^* \tilde{b} - \mathrm{d}\theta \qquad \text{and} \qquad \tilde{q} \mapsto \varphi^* \tilde{q}. \tag{29}$$

Remark 4. Notice also that by construction of $\varphi = \mathrm{Id} + O(\varepsilon_0)$ in C^{k-2}; therefore, the metric $\varphi_* \tilde{g}$ also satisfies (5) with k replaced by $k+3$, and for some $\varepsilon_0' > 0$, such that $\varepsilon_0 \to 0$ as $\varepsilon_0' \to 0$. The same follow for $\varphi^* \tilde{b} - \mathrm{d}\theta$ and $\varphi^* q$, notice that here we are also using that b is close to $\tilde{b}$ as in (5). Hence without loss of generality we can denote $\varphi^* \tilde{g}$, $\varphi^* \tilde{b} - \mathrm{d}\theta$ and $\varphi^* \tilde{q}$ again by $\tilde{g}$, $\tilde{b}$ and $\tilde{q}$. From now on we will follow this notation.

3 Stability at the boundary

We will prove first stability at the boundary following [26]. We consider a highly oscillating solution of (2) asymptotically supported near a single geodesic transversal to $\partial\Omega$. We only need to work locally near a fixed point $x_0 \in \partial\Omega$. Let (x', x^n) be the boundary normal coordinates near x_0. Consider parameters $\lambda \in \mathbb{R}$, $\lambda \gg 1$ and $\omega' \in \mathbb{R}^{n-1}$. Let ε and T' such that $0 < \varepsilon < T' \leq T$. Fix t_0 such that $0 < t_0 < \varepsilon$, and let $\chi \in C_0^\infty(\mathbb{R}_+ \times \partial\Omega)$ be supported in a small enough neighborhood of (t_0, x_0) of radius $\varepsilon' < \min(t_0, \varepsilon - t_0)$ and equals to 1 in a smaller neighborhood of this point. We define u to be the solution of (2) with T' instead of T and

$$f = e^{\mathrm{i}\,\lambda(t - x'\cdot\omega')} \chi(t, x'). \tag{30}$$

One can get an asymtotic expansion for u in a neighborhood $[0, T'] \times U \subset \mathbb{R}^+ \times \Omega$ of (t_0, x_0) by looking for u of the form

$$u = e^{\mathrm{i}\,\lambda(t - \phi(x,\omega))} \sum_{j=0}^{N} \lambda^{-j} A_j(t, x, \omega) + O(\lambda^{-N-1}) \tag{31}$$

in $C^1([0, T']; L^2(U))$, where $N \gg 0$ is fixed and $\omega = (\omega', \omega^n)$ is such that $\sum g^{ij}(x_0)\omega_i\omega_j = 1, \sum g^{ij}(x_0)\nu_i(x_0)\omega_j < 0$. In U the phase function solves the eikonal equation

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial\phi}{\partial x^i} \frac{\partial\phi}{\partial x^j} = 1, \qquad \phi|_{\partial\Omega} = x' \cdot \omega' \tag{32}$$

with the extra condition $\frac{\partial\phi}{\partial\nu}|_{\partial\Omega} < 0$ it is uniquely solvable near x_0. In our coordinates, the metric g satisfies $g_{in} = g^{in} = \delta_{in}$ for $i = 1, \ldots, n$ and $\partial/\partial\nu =$

$-\partial/\partial x^n$. Notice that (32) determines

$$\omega_n(x) = \frac{\partial \phi}{\partial x^n}(x) > 0, \forall x \in U \cap \partial\Omega. \tag{33}$$

In $[0, T'] \times U$ the principal part A_0 of the amplitude solves the first transport equation.

$$LA_0 = 0, \qquad A_0|_{x^n=0} = \chi \tag{34}$$

and the lower order terms solve the transport equation

$$\mathrm{i}\, LA_j = -(\partial_t^2 + P)A_{j-1}, \qquad A_j|_{x^n=0} = 0, \qquad j \geq 1, \tag{35}$$

where

$$L = 2\partial_t + 2\sum_{i,j=1}^{n} g^{ij}\frac{\partial \phi}{\partial x^j}\frac{\partial}{\partial x^i} - 2\mathrm{i}\sum_{i,j=1}^{n} g^{ij} b_i \frac{\partial \phi}{\partial x^j} + \Delta_g \phi. \tag{36}$$

The construction of u guarantees that A_j, $j = 1, \ldots, N$ are supported in a small neighborhood, depending on the size of sup χ, of the characteristics issued from (t_0, x_0) in the codirection $(1, \omega)$. Therefore by the way we choose ε', the term

$$u_N := e^{\mathrm{i}\,\lambda(t-\phi)} \sum_{j=0}^{N} \lambda^{-j} A_j$$

in (31) satisfies the zero initial condition in (2). Moreover, u_N satisfies the boundary condition $u_N = f$ with f as in (30), provided that T' is such that $0 < T' - t_0$, is small enough. Write

$$u = u_N + w.$$

Then $w = w_t = 0$ for $t = 0$, and $w|_{(0,T')\times\partial\Omega} = 0$ and $(\partial_t^2 + P)w = e^{\mathrm{i}\,\lambda(t-\phi)}\lambda^{-N}(\partial_t^2 + P)A_N$. By Lemma (2) we obtain (31) with the estimate remainder in the $C^1([0, T']; L^2(U))$ norm. We emphasize that it is important that $T' - t_0$ is small enough so that the wave does not meet $\partial\Omega$ again (if it does, we need to reflect it off the boundary, as in next section).

In order to get stability at the boundary we are going to take advantage of the freedom that we have of choosing the initial codirection $\omega \in \mathbb{R}^n$ of the solution u. For that we need the following lemma, we postpone its proof to the end of the section.

Lemma 4. *Let M be a Riemannian n-compact manifold with continuous metric g. For any $x_0 \in M$ and any chart (W, ψ) containing x_0 there exist $U \subset W$ open neighborhood of x_0 and $\omega_1(x), \ldots, \omega_N(x) \in S_x^r M = \{\xi \in T_x M : |\xi|_g = r\}$ for all $x \in U$, such that any symmetric 2-tensor field $h(x)$, and any 1-tensor field $\beta(x)$ there exist a constant $C > 0$ such that*

$$\max_{1\leq i,j\leq n} |h_{ij}(x)| + |\beta_j(x)| \leq C \max_{1\leq i\leq N} |T_{\omega_i}[h(x), \beta(x)]| \text{ for all } x \in V$$

where $N = n(n+2)/2$ and $T_{\omega(x)}[h(x), \beta(x)] = h_{ij}(x)\omega^i(x)\omega^j(x) + \beta_j(x)\omega^j(x)$.

The main result of this section is the following:

Theorem 5. *For any $\mu < 1$, $m \geq 0$, there exist $k \gg 1$, such that for any $A > 0$, if*

$$g, \tilde{g}, b, \tilde{b}, q, \tilde{q} \text{ are bounded in } C^k(\bar{\Omega}) \text{ by } A,$$

then there $\exists C > 0$ depending in A and Ω, such that

(i) $\sup_{x \in \partial\Omega, |\gamma| \leq m} |\partial^\gamma (g - \varphi^* \tilde{g})| \leq C ||\Lambda_P - \Lambda_{\tilde{P}}||_*^{\mu/2^m}$

(ii) $\sup_{x \in \partial\Omega, |\gamma| \leq m} |\partial^\gamma (b - (\varphi^* \tilde{b} - d\theta))| \leq C ||\Lambda_P - \Lambda_{\tilde{P}}||_*^{\mu/2^{m+1}}$

(iii) $\sup_{x \in \partial\Omega, |\gamma| \leq m} |\partial^\gamma (q - \varphi^* \tilde{q})| \leq C ||\Lambda_P - \Lambda_{\tilde{P}}||_*^{\mu/2^{m+2}}$

where φ and θ are as in (26) and (27), respectively. C^k norms are taken as in Remark 1.

Proof. By Remark 4, without loss of generality, we denote $\varphi^* \tilde{g}$, $\varphi^* \tilde{b} - \mathrm{d}\theta$ and $\varphi^* \tilde{q}$ again by $\tilde{g}$, $\tilde{b}$ and $\tilde{q}$. To simplify notation let $\delta = ||\Lambda_P - \Lambda_{\tilde{P}}||_*$. In this proof we will denote C by various constants depending only on Ω, A and the choice of $k \gg 1$ in Theorem 1. Using local boundary normal coordinates as before, let $I \times V \subset \mathbb{R}_+ \times \partial\Omega$ be a neighborhood of $t = t_0$, $(x', x^n) = (0, 0)$, where $\chi = 1$. Since $\frac{\partial}{\partial\nu} = -\frac{\partial}{\partial x^n}$ and by Lemma 2, we have

$$\Lambda_P f = e^{i\lambda(t - x\cdot\omega)} \left(i\lambda \frac{\partial \phi}{\partial x^n} - \sum_{j=0}^{N} \lambda^{-j} \left(\frac{\partial}{\partial x^n} - i b_n \right) A_j \right) + O(\lambda^{-N-1}) \tag{37}$$

in $L^2(I \times V)$; similarly for $\Lambda_{\tilde{P}} f$. Then

$$\begin{aligned} \frac{\partial \phi}{\partial x^n} - \frac{\partial \tilde{\phi}}{\partial x^n} = {} & \frac{1}{i\lambda} e^{-i\lambda(t - x\cdot\omega)} (\Lambda_P f - \Lambda_{\tilde{P}} f \\ & + \frac{1}{i\lambda} \left(\sum_{j=0}^{N} \lambda^{-j} \left(\frac{\partial A_j}{\partial x^n} - \frac{\partial \tilde{A}_j}{\partial x^n} \right) - i\lambda^{-j} (b_j A_j - \tilde{b_j} \tilde{A}_j) \right) + O(\lambda^{-N-2}) \end{aligned}$$

in $L^2(I \times V)$. Hence,

$$\left|\left| \frac{\partial \phi}{\partial x^n} - \frac{\partial \tilde{\phi}}{\partial x^n} \right|\right|_{L^2(V)} \leq \frac{C}{\lambda} \left(\delta ||f||_{H^1([0,\varepsilon] \times \partial\Omega)} \right) + \frac{C}{\lambda}$$

Notice that $||f||_{H^1([0,\varepsilon] \times \partial\Omega)} \leq C\lambda$. Now take $\lambda \to \infty$ above to get

$$\left|\left| \frac{\partial \phi}{\partial x^n} - \frac{\partial \tilde{\phi}}{\partial x^n} \right|\right|_{L^2(V)} \leq C\delta. \tag{38}$$

By the eikonal equation (32), in $V \subset \partial\Omega$, we have

$$\omega_n(x) = \frac{\partial \phi}{\partial x^n} = \left(1 - \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x) \omega_\alpha \omega_\beta \right)^{1/2}, \tag{39}$$

and similarly for $\tilde{\omega}_n = \partial\tilde{\phi}/\partial x^n$. Since $||(\omega_n)^2 - (\tilde{\omega}_n)^2||_{L^2(V)} = O(\delta)$ by (38). Then we get

$$||(g^{\alpha\beta}(x) - \tilde{g}^{\alpha\beta}(x)) \omega_\alpha \omega_\beta||_{L^2(V)} = O(\delta).$$

and by Lemma 4 we have

$$||g - \tilde{g}||_{L^2(V)} \leq C\delta. \tag{40}$$

Using compactness the manifold we extend (40) to the whole $\partial\Omega$ hence:

$$||g - \tilde{g}||_{L^2(\partial\Omega)} \leq C\delta. \tag{41}$$

We use interpolation estimates in Sobolev spaces and Sobolev embedding theorems, to get that for any $m \geq 0$ and $\mu < 1$

$$\|g - \tilde{g}\|_{C^m(\partial\Omega)} \leq C\delta^\mu, \tag{42}$$

provided that $k \gg 1$.

To estimate the difference of the first normal derivatives of g and $\tilde{g}$ and the difference between b and $\tilde{b}$ we use up to the principal part in (37), as before

$$\frac{\partial A_0}{\partial x^n} - \frac{\partial \tilde{A}_0}{\partial x^n} - i(b_n A_0 - \tilde{b}_n \tilde{A}_0) = -e^{-i\lambda(t-x\cdot\omega)}(\Lambda_P(u) - \Lambda_{\tilde{P}}(\tilde{u})) + i\lambda\left(\frac{\partial\phi}{\partial x^n} - \frac{\partial\tilde{\phi}}{\partial x^n}\right)$$
$$-\sum_{j=1}^N \lambda^{-j}\left(\frac{\partial A_j}{\partial x^n} - \frac{\partial \tilde{A}_j}{\partial x^n}\right) + \sum_{j=1}^N i\lambda^{-j}(b_n A_j - \tilde{b}_n \tilde{A}_j) + O(\lambda^{-N-1})$$

to obtain

$$\left\|\frac{\partial A_0}{\partial x^n} - \frac{\partial \tilde{A}_0}{\partial x^n} - i(b_n A_0 - \tilde{b}_n \tilde{A}_0)\right\|_{L^2(V)} \leq C\left(\lambda\delta + \delta + \lambda^{-1}\right). \tag{43}$$

The r.h.s above is minimized when $\lambda = \delta^{-1/2}$

$$\left\|\frac{\partial A_0}{\partial x^n} - \frac{\partial \tilde{A}_0}{\partial x^n} - i(b_n A_0 - \tilde{b}_n \tilde{A}_0)\right\|_{L^2(V)} \leq C\delta^{1/2}. \tag{44}$$

The transport equation on $(t_0 - \varepsilon_1, t_0 + \varepsilon_1) \times V$ implies

$$2\omega_n \frac{\partial A_0}{\partial x^n} + \Delta_g \phi - 2i \sum_{i,j=0}^n g^{ij} b_i \omega_j = 0.$$

Assuming the $\omega_n, \tilde{\omega}_n > \eta > 0$ for small η then by (32) and since $g^{in} = \delta_{in}$,

$$\begin{aligned}
2\omega_n \frac{\partial A_0}{\partial x^n} - 2i\omega_n b_n &= \frac{\omega_n}{2\det g}\frac{\partial \det g}{\partial x^n} + \frac{\partial^2\phi}{\partial^2 x^n} - 2ib^\beta\omega_\beta + R \\
&= \frac{\omega_n}{2\det g}\frac{\partial \det g}{\partial x^n} + \frac{1}{2\omega_n}\frac{\partial g^{\alpha\beta}}{\partial x^n}\omega_\alpha\omega_\beta - 2ib^\beta\omega_\beta + R,
\end{aligned}$$

where R involves tangential derivatives of g that we can estimate by (41) and $\partial\phi/\partial x^n$ that we can estimate by (38). Therefore by (44),

$$\begin{aligned}
&\frac{\omega_n}{2\det g}\frac{\partial \det g}{\partial x^n} - \frac{\tilde{\omega}_n}{2\det \tilde{g}}\frac{\partial \det \tilde{g}}{\partial x^n} + \\
&\quad + \frac{1}{2\omega_n}\frac{\partial g^{\alpha,\beta}}{\partial x^n}\omega_\alpha\omega_\beta - \frac{1}{2\tilde{\omega}_n}\frac{\partial \tilde{g}^{\alpha,\beta}}{\partial x^n}\omega_\alpha\omega_\beta + \\
&\quad -2i(b^\beta\omega_\beta - \tilde{b}^\beta\omega_\beta) = O(\delta^{1/2})
\end{aligned} \tag{45}$$

in $L^2(V)$ for all ω's as above. Setting $\omega' = 0$, we get

$$\left|\left|\frac{1}{2\det g}\frac{\partial \det g}{\partial x^n} - \frac{1}{2\det \tilde{g}}\frac{\partial \det \tilde{g}}{\partial x^n}\right|\right|_{L^2(V)} = O(\delta^{1/2}),$$

and then since we can estimate the difference of the metric by (41) we obtain

$$\left|\left|\frac{\partial \det g}{\partial x^n} - \frac{\partial \det \tilde{g}}{\partial x^n}\right|\right|_{L^2(V)} = O(\delta^{1/2}).$$

This last equation together with (45) implies

$$\left|\left|\frac{1}{2\omega_n}\frac{\partial g^{\alpha,\beta}}{\partial x^n}\omega_\alpha\omega_\beta - \frac{1}{2\tilde{\omega}_n}\frac{\partial \tilde{g}^{\alpha,\beta}}{\partial x^n}\omega_\alpha\omega_\beta - 2i(b^\beta\omega_\beta - \tilde{b}^\beta\omega_\beta)\right|\right|_{L^2(V)} = O(\delta^{1/2}),$$

since $\omega_n, \tilde{\omega}_n > \eta > 0$ and using again (38) we have

$$\left|\left|\left(\frac{\partial g^{\alpha,\beta}}{\partial x^n} - \frac{\partial \tilde{g}^{\alpha,\beta}}{\partial x^n}\right)\omega_\alpha\omega_\beta - (4\omega_n i b^\beta - 4\omega_n i\tilde{b}^\beta)\omega_\beta\right|\right|_{L^2(V)} = O(\delta^{1/2}),$$

Let us now take $\omega_n^2 = \tilde{\omega}_n^2 = 1/2$, then

$$\left|\left|\left(\tfrac{\partial g^{\alpha,\beta}}{\partial x^n} - \tfrac{\partial \tilde{g}^{\alpha,\beta}}{\partial x^n}\right)\omega_\alpha\omega_\beta - (ib^\beta - i\tilde{b}^\beta)\omega_\beta\right|\right|_{L^2(V)} = O(\delta^{1/2}).$$

and also w' belongs to $\{\omega' \in T_{x'}\partial\Omega : |\omega'|_{g'} = 1\}$ where g' is the induced metric of either g or $\tilde{g}$ to $\partial\Omega$. Using now Lemma 4 and the fact that near the boundary $b_n - \tilde{b}_n = 0$, see (28), we have

$$||b - \tilde{b}||_{L^2(V)} + \left|\left|\frac{\partial g}{\partial x^n} - \frac{\partial \tilde{g}}{\partial x^n}\right|\right|_{L^2(V)} \le C\delta^{1/2}.$$

Again by compactness we get,

$$||b - \tilde{b}||_{L^2(\partial\Omega)} + \left|\left|\frac{\partial g}{\partial x^n} - \frac{\partial \tilde{g}}{\partial x^n}\right|\right|_{L^2(\partial\Omega)} \le C\delta^{1/2}. \tag{46}$$

As before using interpolation and Sobolev embeddings theorems

$$\left\|\frac{\partial}{\partial x^n}(g - \tilde{g})\right\|_{C^m(\partial\Omega)} + \left\|b - \tilde{b}\right\|_{C^m(\partial\Omega)} \le C\delta^{\mu/2} \tag{47}$$

for any $m \ge 0$ and $\mu < 1$ as long as $k \gg 1$.

To estimate the difference of the second normal derivatives of g and $\tilde{g}$, first normal derivatives of b and $\tilde{b}$ and the difference of q and $\tilde{q}$ we use (37) up to the λ^{-1} to get

$$\left\|\frac{\partial A_1}{\partial x^n} - \frac{\partial \tilde{A}_1}{\partial x^n}\right\|_{L^2(V)} \le C\left(\lambda^2\delta + \lambda\delta^{1/2} + \lambda^{-1}\right).$$

Choose $\lambda = \delta^{-1/4}$ to obtain

$$\left\|\frac{\partial A_1}{\partial x^n} - \frac{\partial \tilde{A}_1}{\partial x^n}\right\|_{L^2(V)} \le C\delta^{-1/4}. \tag{48}$$

Using the equation $iLA_i = -(\partial_t^2 + P)A_0$ restricted to $I \times V$ we get

$$2i\omega_n \frac{\partial A_1}{\partial x^n} = \frac{1}{2\det g}\frac{\partial \det g}{\partial x^n}\frac{\partial A_0}{\partial x^n} + \frac{\partial^2 A_0}{\partial^2 x^n} + q,$$

this together with (38), (41) and (48) gives

$$\left|\left|2\omega_n \frac{\partial^2 A_0}{\partial^2 x^n} - 2\tilde{\omega}_n \frac{\partial^2 \tilde{A}_0}{\partial^2 x^n} + 2\omega_n q - 2\tilde{\omega}_n \tilde{q}\right|\right|_{L^2(V)} = O(\delta^{1/4}). \tag{49}$$

Assuming $\omega_n, \tilde{\omega}_n > \eta > 0$ for small η and taking normal derivatives in (34) and restricting it to $I \times V$ we have

$$\begin{aligned} 2\omega_n \frac{\partial^2 A_0}{\partial^2 x^n} &= \frac{\omega_n}{2\det g}\frac{\partial^2 \det g}{\partial^2 x^n} + \frac{\partial^3 \phi}{\partial^3 x^n} - 2ig^{\alpha\beta}\frac{\partial b_\alpha}{\partial x^n}\omega_\beta + R \\ &= \frac{\omega_n}{2\det g}\frac{\partial^2 \det g}{\partial^2 x^n} + \frac{1}{2\omega_n}\frac{\partial^2 g^{\alpha\beta}}{\partial^2 x^n}\omega_\alpha\omega_\beta - 2ig^{\alpha\beta}\frac{\partial b_\alpha}{\partial x^n}\omega_\beta + R \end{aligned}$$

where R consist in term that we can estimate by (41) and (46). Hence we get

$$\begin{aligned} &\frac{\omega_n}{2\det g}\frac{\partial^2 \det g}{\partial^2 x^n} - \frac{\tilde{\omega}_n}{2\det \tilde{g}}\frac{\partial^2 \det \tilde{g}}{\partial^2 x^n} + 2\omega_n q - 2\tilde{\omega}_n \tilde{q} + \\ &\qquad + \frac{1}{2\omega_n}\frac{\partial^2 g^{\alpha,\beta}}{\partial^2 x^n}\omega_\alpha\omega_\beta - \frac{1}{2\tilde{\omega}_n}\frac{\partial^2 \tilde{g}^{\alpha,\beta}}{\partial^2 x^n}\omega_\alpha\omega_\beta + \\ &\qquad - 2i\left(g^{\alpha\beta}\frac{\partial b_\alpha}{\partial x^n}\omega_\beta - \tilde{g}^{\alpha\beta}\frac{\partial \tilde{b}_\alpha}{\partial x^n}\omega_\beta\right) = O(\delta^{1/4}) \end{aligned}$$

in $L^2(V)$. Again, setting $\omega' = 0$ we have

$$\left|\left|\frac{1}{2\det g}\frac{\partial^2 \det g}{\partial^2 x^n} - \frac{1}{2\det \tilde{g}}\frac{\partial^2 \det \tilde{g}}{\partial^2 x^n} + 2q - 2\tilde{q}\right|\right|_{L^2(V)} = O(\delta^{1/4}). \tag{50}$$

Now since $\omega_n - \tilde{\omega}_n = O(\delta)$ then

$$\left|\left|\frac{1}{2\omega_n}\left(\frac{\partial^2 g^{\alpha,\beta}}{\partial^2 x^n} - \frac{\partial^2 \tilde{g}^{\alpha,\beta}}{\partial^2 x^n}\right)\omega_\alpha\omega_\beta - 2i\left(g^{\alpha\beta}\frac{\partial b_\alpha}{\partial x^n} - \tilde{g}^{\alpha\beta}\frac{\partial \tilde{b}_\alpha}{\partial x^n}\right)\omega_\beta\right|\right|_{L^2(V)} = O(\delta^{1/4}).$$

We use similar reasoning as before. First we use Lemma 4 and then compactness to get that $\left|\left|\frac{\partial b}{\partial x^n} - \frac{\partial \tilde{b}}{\partial x^n}\right|\right|_{L^2(\partial\Omega)}$ and $\left|\left|\frac{\partial^2 g}{\partial^2 x^n} - \frac{\partial^2 \tilde{g}}{\partial^2 x^n}\right|\right|_{L^2(\partial\Omega)}$ are $O(\delta^{1/4})$, this together with (50) gives

$$||q - \tilde{q}||_{L^2(\partial\Omega)} + \left|\left|\frac{\partial b}{\partial x^n} - \frac{\partial \tilde{b}}{\partial x^n}\right|\right|_{L^2(\partial\Omega)} + \left|\left|\frac{\partial^2 g}{\partial^2 x^n} - \frac{\partial^2 \tilde{g}}{\partial^2 x^n}\right|\right|_{L^2(\partial\Omega)} \leq C\delta^{1/4}. \tag{51}$$

As before we get

$$||q - \tilde{q}||_{C^m(\partial\Omega)}, \left|\left|\frac{\partial b}{\partial x^n} - \frac{\partial \tilde{b}}{\partial x^n}\right|\right|_{C^m(\partial\Omega)}, \left|\left|\frac{\partial^2 g}{\partial^2 x^n} - \frac{\partial^2 \tilde{g}}{\partial^2 x^n}\right|\right|_{C^m(\partial\Omega)} \leq C\delta^{\mu/4}. \tag{52}$$

for $m > 0$ and $\mu < 1$. Proceeding by induction we prove the theorem. □

Proof of Lemma 4. First notice that by rescaling it is enough to proof the theorem for $r = 1$. Let $\xi_1(x_0), \dots, \xi_N(x_0)$ be like in Lemma 3.3 in [8] related to x_0. Consider

$$\omega_m^i(x) = \frac{\xi_m^i(x_0)}{|\xi_m(x_0)|_g} \;\forall i = 1, \dots, n \text{ and } m = 1, \dots, N.$$

By continuity of the metric, for any $\epsilon > 0$, there exist a small neighborhood U of x_0 were $|\xi_m(x_0)|_g > 1/2$ and $\sup_{x\in V} \max_{m,i} |\omega_m^i(x) - \xi_m^i(x_0)| < \epsilon$. Now notice that in Lemma 3.3 in [8] the determination of $[h, \beta]$ is done by inverting any of the linear transformation $T_\Omega : [h, \beta] \mapsto (T_{\omega_1}[h, \beta], \dots, T_{\omega_N}[h, \beta])$, whose inverse is also linear. Choose ϵ small enough so that linear independence is preserved, then we can take $C = \sup_{x\in U} ||T_{\Omega(x)}^{-1}||$, were U might be a smaller neighborhood of x_0. □

4 Interior Stability

We first estimate the difference in the metrics and its derivatives following the argument in [26] that is based on the boundary rigidity result in[25]. We then estimate the difference in the co-vector fields and the potentials using semigeodesical coordinates related to a point $z_0 \notin \bar{\Omega}$ but close enough so that simplicity assumption is still valid. We use the phase function

$$\phi = x^n = d(x, z_0).$$

Then the transport operator (36) becomes

$$L = 2\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x^n} - 2ib_n + \frac{1}{\sqrt{\det g}}\frac{\partial \det g}{\partial x^n}, \tag{53}$$

and can be solved explicitly. We get non-weighted integral transforms of b and q. We then apply Hölder stability theorem in [24] for vector fields and functions to prove the theorem.

For technical reasons we will need to make a δ-small perturbation of $\tilde{g}, \tilde{b}$ and $\tilde{q}$ so that near the boundary they coincide with g, b and q. We will follow notation as in Remark 4.

By Theorem 5, one has that for any $m > 0$, there exist $0 < \mu < 1$ and $k \gg 1$, such that

$$\sup_{x\in\partial\Omega, |\gamma|\le m} |\partial^\gamma(g - \tilde{g})| \le C\delta^\mu. \tag{54}$$

Let $m > 0$ be any integer and let $\chi \in C^\infty(\mathbb{R})$, $\chi = 1$ for $t < 1$ and $\chi = 0$ for $t > 2$. Set

$$\tilde{g}_1 = \tilde{g} + \chi\left(\delta^{-1/M}\rho(x, \partial\Omega)\right)(g - \tilde{g}), \tag{55}$$

where $M = 2m/\mu$ Using a finite Taylor expansion of $g - \tilde{g}$ up to $O((x^n)^M)$, where $x^n = \rho(x, \partial\Omega)$ and estimate (54), we see that

$$||\tilde{g}_1 - \tilde{g}||_{C^m(\overline{\Omega})} \le C\delta^{\mu - m/M} = C\delta^{\mu/2} \tag{56}$$

Is important to notice that $\tilde{g}_1$ depends on m, but we will only need (55) for large fixed m. In particular the estimate above implies that $\tilde{g}_1$ is also simple for $\delta \ll 1$. As before without loss of generality we can assume that (23) are still true for g and $\tilde{g}_1$. We extend g and $\tilde{g}_1$ in the same way as simple metrics in a neighborhood $\Omega_1 \supset \overline{\Omega}$. The advantage we have now is that

$$g = \tilde{g}_1 \text{ for } -1/C \leq \rho(x, \partial\Omega) \leq \delta^{1/M}. \tag{57}$$

and hence by strictly convexity of the boundary there exist a constant C_κ such that

$$(\rho_g - \rho_{\tilde{g}_1})|_{\partial\Omega\times\Omega} = 0 \text{ if } \rho_g(x,y) \leq C_\kappa \delta^{1/(2M)}, \tag{58}$$

where ρ_g denotes the distance function with respect to g. Moreover, using (56) we obtain

$$|\rho_{\tilde{g}}(x,y) - \rho_{\tilde{g}_1}(x,y)| \leq C\delta^{\mu/2} \ \ \forall x, y \in \overline{\Omega}. \tag{59}$$

We define similarly

$$\begin{aligned} \tilde{b}_1 &= \tilde{b} + \chi\left(\delta^{-1/M}\rho(x,\partial\Omega)\right)(b - \tilde{b}) \\ \tilde{q}_1 &= \tilde{q} + \chi\left(\delta^{-1/M}\rho(x,\partial\Omega)\right)(q - \tilde{q}) \end{aligned} \tag{60}$$

and use Theorem 5 to get that

$$||\tilde{b}_1 - \tilde{b}||_{C^m(\overline{\Omega})} + ||\tilde{q}_1 - \tilde{q}||_{C^m(\overline{\Omega})} \leq C\delta^{1/(2M)} \tag{61}$$

and

$$b = \tilde{b}_1 \text{ and } q = \tilde{q}_1 \text{ for } -1/C \leq \rho(x,\partial\Omega) \leq \delta^{1/M} \tag{62}$$

We proceed to the proof of the Main Theorem:

Proof of Theorem 1. Recall the notation $\delta = ||\Lambda - \tilde{\Lambda}||_*$. It is enough to prove the proposition for $\delta \ll 1$. We use notation as in Remark 4. In what follows we denote $\mu < 1$ constants arbitrarily close to 1 that might change from step to step. We also denote by C various constants depending only in Ω, A and the choice of k in Theorem 1.

We first modify the mertic $\tilde{g}, \tilde{b}$ and $\tilde{q}$ by $\tilde{g}_1, \tilde{b}_1$ and $\tilde{q}_1$ as in (55) and (60). We denote this extensions again by the same notation. From now on objects below related to g are without tildes and those related to $\tilde{g}_1$ are with a tilde above (no subscript 1). Fix $x_0, y_0 \in \partial\Omega$. We will construct an oscillating solution related to g similarly to the one used in the previous section, but in this case we want it to go all the way from x_0 to y_0. Consider the geodesic from x_0 to y_0, extended from Ω to $z_0 \in \Omega_1$ such that the geodesic segment $[z_0, x_0] \subset \Omega_1 \backslash \Omega$, see Figure 2. Assume also that $\rho_g(z_0, x_0) > 1/C > 0$.

Consider (x', x^n) global semi-geodesic coordinates related to z_0, where x^n is the distance from z_0 to the point p and x' is a parametrization of the angular variable so that $x' = $ const. are geodesics issued from z_0 with x^n as the arc-lenght parameter, as in Lemma 4.2 in [25]. We have in this coordinates

$$x^n = \rho_g(p, z_0) \qquad \text{and} \qquad g_{in} = \delta_{in}.$$

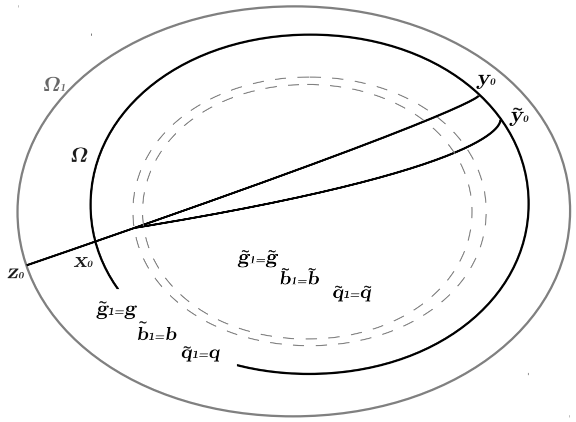

Figure 2: Solutions near semigeodesical coordinates.

The prove is divided in three step:
(a) Metric Stability: We follow [26] and use boundary rigidity estimates in [25]. Some simplifications to the argument are presented. We claim that

$$||\rho_g - \rho_{\tilde{g}_1}||_{C(\partial\Omega\times\partial\Omega)} \leq C\delta^{\mu} \text{ for } 0 < \mu < 1. \tag{63}$$

Assuming the claim and using estimate (59) we have

$$||\rho_g - \rho_{\tilde{g}}||_{C(\partial\Omega\times\partial\Omega)} \leq C\delta^{\mu} \text{ for } 0 < \mu < 1.$$

We then apply estimate in Theorem 1.8 in [25] to get

$$||g - \tilde{g}||_{C^2(\overline{\Omega})} \leq C\delta^{\mu} \text{ for } 0 < \mu < 1. \tag{64}$$

We will sketch the proof of the claim (63) since is similar to that in [26]. We write the explicit construction as in there since it will be used in part (b) and (c) of the proof. In view of (58), to prove (63) we assume w.l.o.g. that $\rho(x_0, y_0) \geq C_\kappa \delta^{\mu'}$. We claim that

$$|\rho_g^2(x_0, y_0) - \rho_{\tilde{g}}^2(x_0, y_0)| = O(\delta^{\mu}) \tag{65}$$

with some $0 < \mu < 1$ uniformly w.r.t. x_0 and y_0, which implies (63).

Let $T > \operatorname{diam}\Omega$, w.l.o.g. assume that $T - \operatorname{diam}\Omega$ is small, see (13). For $0 < t_0 \ll 1$ define

$$U = \left\{(t, x) \in \mathbb{R}_+ \times \partial\Omega; |t - t_0| + \rho(x, x_0) < \delta^{1/(2M)}/C_1\right\} \tag{66}$$

where $C_1 \gg 1$ will be specified later. Choose a cut-off function $0 \leq \chi \leq 1, \chi \in C_0^\infty(\mathbb{R}_+ \times \partial\Omega)$ such that $\sup \chi \subset U$, and $\chi = 1$ in a set defined as U but with C_1 replaced by $2C_1$. One can arrange that $|\partial_t \chi| + |\nabla\chi| \leq C\delta^{-1/(2M)}$.

Set $\phi(x) = \rho(x, z_0)$. Then, by simplicity assumption, since $z_0 \in \Omega_1$, we have that $\phi \in C^{k-1}(\overline{\Omega})$, and ϕ solves que eikonal equation

$$\sum_{i,j=1}^{\infty} g^{ij} \frac{\partial\phi}{\partial x^i} \frac{\partial\phi}{\partial x^j} = 1.$$

Now we construct a solution u of (2) for

$$f = e^{-\mathrm{i}\,\lambda(t-\phi)}\chi.$$

There is a need of reflecting the solution ones it reaches the boundary to get the zero boundary condition, so the solution u is the sum of the projected wave and the reflected wave, $u = u_P + u_R$ with

$$u_I(t,x;\lambda) = e^{-\mathrm{i}\,\lambda(t-\phi(x))}(A_0(t,x) + \lambda^{-1}A_1(t,x) + R(t,x;\lambda)),$$
$$u_R(t,x;\lambda) = e^{-\mathrm{i}\,\lambda(t-\hat{\phi}(x))}(\hat{A}_0(t,x) + \lambda^{-1}\hat{A}_1(t,x) + \hat{R}(t,x;\lambda)).$$

where

$$||R||_{C^1([0,T];L^2(\Omega))} + ||\hat{R}||_{C^1([0,T];L^2(\Omega))} \le \frac{C}{\lambda^2}, \tag{67}$$

This solutions satisfy the following equations:

$$LA_0 = 0, \;\; A_0|_U = \chi; \quad \mathrm{i}\,LA_1 = -(\partial_t^2 + P)A_0, \;\; A_j|_U = 0 \tag{68}$$

$$L\hat{A}_0 = 0, \;\; \hat{A}_0|_V = -A_0|_V; \quad \mathrm{i}\,L\hat{A}_1 = -(\partial_t^2 + P)\hat{A}_0, \;\; \hat{A}_1|_V = -A_1|_V \tag{69}$$

where $V \subset \mathbb{R}_+ \times \partial\Omega$ is the image of U under translations by all geodesics issued from z_0 and passing through U. The phase function $\hat{\phi}$ still solves the eikonal equation with boundary condition $\hat{\phi}|_V = \phi$ and is unique solution with gradient pointing towards the interior of Ω (the opposite solution in ϕ). All this boundary conditions are assumed to be extended as zero in the rest of the boundary.

We can solve the solve the transport equations (68) and (69) in a neighborhood of the geodesic connecting x_0 and y_0 of size $O(\delta^{1/(2M)})$, and by simplicity assumption, this solutions can be extended all the way to V. If C_1 in (66)is large enough, then U and V are disjoint sets

Because of the strict convexity of $\partial\Omega$, each component is of size $O(\delta^{1/(2M)})$, at a distance bounded from below by the same quantity by assumption. Denote by $B(y,r)$ the ball centered at y with radius r. Then V contains the set $V_0 = V \cap B(y_0, \delta^{1/(2M)}/C^0)$, such that on V_0, we have $A_0 \ge 1/C > 0$. Above, C^0 is chosen so that V_0 is contained in the translation of the set $\{\chi = 1\}$ under geodesics issued from z_0.

Next, we construct a similar solution $\tilde{u}$ related to $\tilde{g}$. We construct first a phase function $\tilde{\phi}$ as $\tilde{\phi} = \tilde{\rho}(x, z_0)$. It solves the eikonal equation

$$\sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{\partial\tilde{\phi}}{\partial x^i}\frac{\partial\tilde{\phi}}{\partial x^j} = 1, \quad \tilde{\phi}|_U = \phi \tag{70}$$

The later equality follows from (57). The other properties of $\tilde{u}$ are similar to those of u. Let $\tilde{V}_0$ be defined as above, but associated to $\tilde{g}$.

We claim that $V_0 \cap \tilde{V}_0 \ne \emptyset$ arguing by contradiction. Notice that

$$||f||^2_{H^1([0,T]\times\partial\Omega)} \le C\delta^{n/(2M)}\lambda^2 + C(\delta) \tag{71}$$

where the first constant is independent of δ (but it depends on A and ϵ_0 in (23)). To prove this, we only need to estimate $A_0\nabla\phi$ on U, and use that $|\partial_t\chi| + |\nabla\chi| \leq C\delta^{-1/(2M)}$. On V_0, we have

$$\Lambda_P f = -2\mathrm{i}\,\lambda e^{\mathrm{i}\,\lambda(\phi - t)} A_0 \frac{\partial\phi}{\partial x^n} + O_\delta(1) \tag{72}$$

in $L^2(V_0)$ and similarly for $\Lambda_{\tilde{P}} f$. Notice that

$$\left|\left|\frac{\partial\phi}{\partial x^n} A_0\right|\right|^2_{L^2(V_0)} \geq \delta^{n/(2M)}/C \tag{73}$$

because area $(V_0) \geq \delta^{n/(2M)}/C$. Since V_0 and $\tilde{V}_0$ do not intersect and by (71) we obtain

$$\lambda^2\delta^{n/(2M)}/C - C(\delta) \leq ||\Lambda_P f - \Lambda_{\tilde{P}} f||^2_{L^2[0,T]\times\partial\Omega} \leq C\lambda^2\delta^{1+n/(2M)} + C(\delta).$$

Dividing last inequality by λ^2 and taking $\lambda \to \infty$ we get a contradiction. Hence $q \in V_0 \cap \tilde{V}_0$, using simplicity assumption and triangular inequality we get (65) and this concludes section (a).

(b) Magnetic Field Stability: For this part we will use the stability of the principal part in the solution constructed in (a) and stability of the 1-tensor geodesic x-ray transform. We will use sharp estimates of the 1-tensor x-ray transform obtained in [24]. Stability for this 1-tensor geodesic x-ray transform was previously know, see [21] and references in there. As before

$$||f||^2_{H^1([0,T]\times\partial\Omega)} \leq C\lambda^2 + C(\delta)$$

where the first constant is independent of δ, but it depends on A and ϵ_0 in (23). We also have

$$||\Lambda_P f - \Lambda_{\tilde{P}} f||^2_{L^2[0,T]\times\partial\Omega} \leq C\lambda^2\delta + C(\delta).$$

Using this two last equations, (72) and part (a) we obtain

$$||A_0 - \tilde{A}_0||_{L^2(V_0\cap\tilde{V}_0)} \leq C\delta + C(\delta)/\lambda,$$

taking $\lambda \to \infty$ we get

$$||A_0 - \tilde{A}_0||_{L^2(V_0\cap\tilde{V}_0)} \leq C\delta \tag{74}$$

In this coordinate system, remember $\phi = x^n$, the transport equation (36) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^n} + \frac{1}{2\sqrt{\det g}}\frac{\partial \det g}{\partial x^n} - ib_n\right) A_0 = 0 \tag{75}$$

with same initial conditions as in (68). Using the method of characteristics or a change of variables we can compute explicitly this solution and get

$$A_0(t,x) = \left(\tfrac{\det g(x',0)}{\det g(x)}\right)^{1/4} e^{\mathrm{i}\int_0^{x^n} b_n(s)ds}\chi(x', t - x^n).$$

If we stay near $t = x^n$, then $\chi = 1$ so

$$A_0(t,x) = \left(\tfrac{\det g(x',0)}{\det g(x)}\right)^{1/4} e^{\mathrm{i}\int_0^{x^n} b_n(s)ds}. \tag{76}$$

Using (74), and (64) we get that

$$||e^{\mathrm{i}\int_0^{x^n} b_n(s)-\tilde{b}_n(0,s)ds} - 1||_{L^2(V_0\cap\tilde{V}_0)} \le C\delta^\mu. \tag{77}$$

Remember that we modified the covector field to that $b-\tilde{b}=0$ in a neighborhood of the boundary containing $\Omega_1\setminus\Omega$, then $b-\tilde{b}$ belongs to $C^1(\overline{\Omega})$ and by (5) and (61) we have

$$||b-\tilde{b}||_{C^1(\bar{\Omega})} \le C(\epsilon_0+\delta^{\mu/2}).$$

Since $\delta \ll 1$ we can use a Taylor expansion of e^{ix} near zero to get

$$\left|\left|\int_0^{x^n} b_n(0,s)-\tilde{b}_n(0,s)ds\right|\right|_{L^2(V_0\cap\tilde{V}_0)} \le C\delta^\mu. \tag{78}$$

This is the coordinate representation of the x-ray transform algon the geodesic starting at z_0 and going all the way to y_0. Until know we had a fixed coordinate system associated to z_0. Since all constants C are uniform with respecto to x_0 and y_0 We can then shoot rays in all directions and we can move z_0 around all $\partial\Omega_1$. We note that since we modify the covector field near the boundary of Ω there are no non-zero tangential rays that are being integrated over. Hence, since $I_g : L^2(\Omega_1)\to L^2(\Gamma_-(\Omega_1);d\mu)$ is bounded by [21] we have

$$||I_g(b-\tilde{b})||_{L^2(\Gamma_-(\Omega_1);d\mu)} \le C\delta^\mu. \tag{79}$$

Using interpolation and the compactness assumption of the metric and the potential we can estimate

$$||I_g(b-\tilde{b})||_{C^1(\Gamma_-(\Omega_1);d\mu)} \le C\delta^\mu.$$

By Theorem 4 in [24] we know

$$||b-\tilde{b}_1||_{L^2(\Omega)} \le C||I_g^*I_g(b-\tilde{b})||_{H^1(\Omega)} \tag{80}$$

with

$$(I_g^*\psi(x))^i = \int_{|\xi|_g=1} \xi^i\psi(\gamma_{x,\xi})d\xi,$$

were $\gamma_{x,\xi}$ denotes the maximal geodesic in Ω_1 that passes through x with codirection ξ. Hence

$$||b-\tilde{b}||_{L^2(\Omega_1)} \le C\delta^\mu \tag{81}$$

for some $0<\mu<1$. Now by (61) and since $\tilde{b}=\tilde{b}_1$ on $\Omega_1\setminus\Omega$ then

$$||b-\tilde{b}||_{L^2(\Omega)} \le C\delta^\mu. \tag{82}$$

By interpolation and compactness we get

$$||b-\tilde{b}||_{C^1(\Omega)} \le C\delta^\mu. \tag{83}$$

(b) Potential Stability: Finally for the potential we use the next term in the expansion of the previous solution and stability estimates for the geodesic x-ray transform of functions in [24]. We have that the DN-map is given by

$$\Lambda_P f = e^{-\mathrm{i}\,\lambda(t-\phi)}\left(-2\mathrm{i}\,\lambda\frac{\partial\phi}{\partial\nu}A_0 - 2\mathrm{i}\,\frac{\partial\phi}{\partial\nu}A_1 + (\partial_\nu A_0 - \partial_\nu\hat{A}_0)\right) + O(\lambda^{-1})$$

in $L^2(V_0)$ and similarly for $\Lambda_{\tilde{P}}$. Notice that since the metrics g and $\tilde{g}$ are equal in a neighborhood of the boundary then all rays can be taken transversal to the boundary. Hence

$$\frac{\partial}{\partial \nu} = a(x)\frac{\partial}{\partial x^n} + c(x)\frac{\partial}{\partial \nu^\perp}$$

with $|a(x)| > 1/C$ in $V_0 \cap \tilde{V}_0$, where $\partial_{\nu^\perp}$ is tangential to the boundary. Since we can estimate tangential derivatives we get

$$-2\mathrm{i}\,\frac{\partial \phi}{\partial \nu}(A_1-\tilde{A}_1)+2a(x)\frac{\partial}{\partial x^n}(A_0-\tilde{A}_0) = \Lambda_P f-\Lambda_{\tilde{P}} f+2\mathrm{i}\, a(x)\lambda(A_0-\tilde{A}_0)+O(\lambda^{-1})+O(\delta^\mu)$$

in $L^2(V_0\cap\tilde{V}_0)$. Now using the explicit form of the amplitude (76) to estimmate $A_0 - \tilde{A}_0$ we obtain

$$\left|\left|-\mathrm{i}\,\frac{\partial \phi}{\partial \nu}(B_1 - \tilde{B}_1)\right|\right|_{L^2(V_0\cap\tilde{V}_0)} \le C(\lambda\delta + 1/\lambda) + O(\delta^\mu),$$

taking $\lambda = \delta^{-1/2}$ we get

$$\left|\left|A_1 - \tilde{A}_1\right|\right|_{L^2(V)} \le C\delta^\mu \tag{84}$$

and

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^n} + \frac{1}{2\sqrt{\det g}}\frac{\partial \det g}{\partial x^n} - ib_n\right) A_1 = (\partial_t^2 + P)A_0. \tag{85}$$

By (9) then

$$(\partial_t^2 + P)A_0 = \partial_t^2 A_0 - \triangle_g A_0 - qA_0 - |b|_g^2 A_0 - \mathrm{i}\ (2\langle b, \mathrm{d}A_0\rangle_g + (\mathrm{Div} b)A_0)\,.$$

Again, near $t = x^n$ we get

$$\begin{aligned} A_1(x,t) = \tfrac{1}{\eta(x,t)} \int_0^{x^n} & [\eta(x',s)\triangle_g A_0 - (\det g(x'))^{1/2} q - (\det g(x'))^{1/4}|b|_g \\ & + \mathrm{i}\, 2\eta(x',s)\,(2\langle b, \mathrm{d}A_0\rangle_g + (\mathrm{Div} b)A_0)]ds \end{aligned}$$

where

$$\eta((x,0)', x^n) = (\det g(x))^{1/4} e^{-\mathrm{i}\int_0^{x^n} b_n(s)ds}$$

Now by (84) and previous estimates we have

$$\left|\left|\tfrac{(\det g(x',0))^{1/4}}{\eta(x,t)}\int_0^{x^n} q_n(s)ds - \tfrac{(\det \tilde{g}(x',0))^{1/4}}{\tilde{\eta}(x,t)}\int_0^{x^n} \tilde{q}_n(s)ds\right|\right|_{L^2(V)} \le C\delta^\mu,$$

since we have estimates for the metrics and the covector field this implies

$$\left|\left|\int_0^{x^n} q_n(s) - \tilde{q}_n(s)ds\right|\right|_{L^2(V)} \le C\delta^\mu.$$

Comparing this with (78), we ague as before, we obtain an invariant representation of the previous inequality

$$||I_g(q-\tilde{q})||_{L^2(\Gamma_-(\Omega_1);d\mu)} \leq C\delta^\mu.$$

Using interpolation and the compactness assumption of the metric and the potential we get estimate (remember we changed the potencial so that $q = \tilde{q}$ near the boundary)

$$||I_g(q-\tilde{q})||_{C^1(\Gamma_-(\Omega_1);d\mu)} \leq C\delta^\mu.$$

We then apply an stability estimate for the geodesic x-ray transform as in Theorem 3 in [24] and interpolation to get

$$||q-\tilde{q}||_{C(\Omega)} \leq C\delta^\mu$$

for some $0 < \mu < 1$. □

References

[1] Giovanni Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), no. 1-3, 153–172. MR 922775 (89f:35195)

[2] Michael Anderson, Atsushi Katsuda, Yaroslav Kurylev, Matti Lassas, and Michael Taylor, *Boundary regularity for the Ricci equation, geometric convergence, and Gel′fand's inverse boundary problem*, Invent. Math. **158** (2004), no. 2, 261–321. MR 2096795 (2005h:53051)

[3] M. I. Belishev, *An approach to multidimensional inverse problems for the wave equation*, Dokl. Akad. Nauk SSSR **297** (1987), no. 3, 524–527. MR 924687 (89c:35152)

[4] ———, *Boundary control in reconstruction of manifolds and metrics (the BC method)*, Inverse Problems **13** (1997), no. 5, R1–R45. MR 1474359 (98k:58073)

[5] Michael I. Belishev and Yaroslav V. Kurylev, *To the reconstruction of a Riemannian manifold via its spectral data (BC-method)*, Comm. Partial Differential Equations **17** (1992), no. 5-6, 767–804. MR 1177292 (94a:58199)

[6] Mourad Bellassoued and David Dos Santos Ferreira, *Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map*, Inverse Problems **26** (2010), no. 12, 125010, 30. MR 2737744 (2012c:58040)

[7] Fernando Cardoso and Ramón Mendoza, *On the hyperbolic Dirichlet to Neumann functional*, Comm. Partial Differential Equations **21** (1996), no. 7-8, 1235–1252. MR 1399197 (97g:35016)

[8] Nurlan S. Dairbekov, Gabriel P. Paternain, Plamen Stefanov, and Gunther Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*, Adv. Math. **216** (2007), no. 2, 535–609. MR 2351370 (2008m:37107)

[9] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Comm. Partial Differential Equations **32** (2007), no. 10-12, 1737–1758. MR 2372486 (2008k:35491)

[10] Gregory Eskin, *Inverse scattering problem in anisotropic media*, Comm. Math. Phys. **199** (1998), no. 2, 471–491. MR 1666879 (2000d:35251)

[11] Victor Isakov, *Inverse problems for partial differential equations*, second ed., Applied Mathematical Sciences, vol. 127, Springer, New York, 2006. MR 2193218 (2006h:35279)

[12] Victor Isakov and Zi Qi Sun, *Stability estimates for hyperbolic inverse problems with local boundary data*, Inverse Problems **8** (1992), no. 2, 193–206. MR 1158175 (93g:35140)

[13] Alexander Katchalov, Yaroslav Kurylev, and Matti Lassas, *Inverse boundary spectral problems*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 123, Chapman & Hall/CRC, Boca Raton, FL, 2001. MR 1889089 (2003e:58045)

[14] Yaroslav Kurylev and Matti Lassas, *Hyperbolic inverse boundary-value problem and time-continuation of the non-stationary Dirichlet-to-Neumann map*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 4, 931–949. MR 1926923 (2003h:35279)

[15] Yaroslav V. Kurylev and Matti Lassas, *The multidimensional Gel′fand inverse problem for non-self-adjoint operators*, Inverse Problems **13** (1997), no. 6, 1495–1501. MR 1484000 (98m:35224)

[16] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl. (9) **65** (1986), no. 2, 149–192. MR 867669 (88c:35092)

[17] Adrian Nachman, John Sylvester, and Gunther Uhlmann, *An n-dimensional Borg-Levinson theorem*, Comm. Math. Phys. **115** (1988), no. 4, 595–605. MR 933457 (89g:35082)

[18] Richard S. Palais, *Extending diffeomorphisms*, Proc. Amer. Math. Soc. **11** (1960), 274–277. MR 0117741 (22 #8515)

[19] Rakesh and William W. Symes, *Uniqueness for an inverse problem for the wave equation*, Comm. Partial Differential Equations **13** (1988), no. 1, 87–96. MR 914815 (89f:35208)

[20] V. G. Romanov, *Uniqueness theorems in inverse problems for some second-order equations*, Dokl. Akad. Nauk SSSR **321** (1991), no. 2, 254–257. MR 1153550 (93f:35241)

[21] V. A. Sharafutdinov, *Integral geometry of tensor fields*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1994. MR 1374572 (97h:53077)

[22] Takahiro Shiota, *An inverse problem for the wave equation with first order perturbation*, Amer. J. Math. **107** (1985), no. 1, 241–251. MR 778095 (86i:35143)

[23] Plamen Stefanov and Gunther Uhlmann, *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media*, J. Funct. Anal. **154** (1998), no. 2, 330–358. MR 1612709 (99f:35120)

[24] ______, *Stability estimates for the X-ray transform of tensor fields and boundary rigidity*, Duke Math. J. **123** (2004), no. 3, 445–467. MR 2068966 (2005h:53130)

[25] ______, *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc. **18** (2005), no. 4, 975–1003 (electronic). MR 2163868 (2006h:53031)

[26] ______, *Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map*, Int. Math. Res. Not. (2005), no. 17, 1047–1061. MR 2145709 (2006a:58030)

[27] Zi Qi Sun, *On continuous dependence for an inverse initial-boundary value problem for the wave equation*, J. Math. Anal. Appl. **150** (1990), no. 1, 188–204. MR 1059582 (91i:35024)

[28] John Sylvester and Gunther Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) **125** (1987), no. 1, 153–169. MR 873380 (88b:35205)

[29] Daniel Tataru, *Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem*, Comm. Partial Differential Equations **20** (1995), no. 5-6, 855–884. MR 1326909 (96e:35019)

[30] Hans Triebel, *Interpolation theory, function spaces, differential operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 1328645 (96f:46001)